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Original article

Direct solution to problems of static sharp waves in shallow-water

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ABSTRACT

The study of hydrodynamic canals is the first step in canals design, sediment transport, erosion, dissemination of pollution and other phenomena related to canals. When in canals, depth or flow rate is suddenly changed, the sharp wave is generated. When the position and characteristics of the sharp wave remain constant, after the steady flow, it is called the static-sharp wave. So generated jump hydraulic and sharp waves in transitions due to cross-obstacle in flow path at supercritical flows are classified as the static sharp waves. Since the equations governing the dynamics of sharp waves are the same as the flow equation in shallow-water, this paper uses linear solve for the equations governing shallow-water and specifying boundary conditions in shallow-water with the static wave and by using Taylor series, an equation for the bottom and two equations based on kinematic and dynamic boundary for the surface boundary were extracted. Also considering the frequency of the surface waves, frequency boundary conditions considering introduction of the dimensionless parameter θ based on wave number and angular frequency, were extracted and other boundary conditions were rewritten based on the dimensionless parameter θ. Next, based on the velocity potential and the Laplace equation, generated the differential equations solved by Euler equation, which leads to generate potential velocity (ϕ) and wave height (η) is the canal length.

1. Introduction

The study of hydrodynamic canals is the first step in canals design, sediment transport, erosion, dissemination of pollution and other phenomena related to canals.

An important part of the hydrodynamic is focusing on to the flow in canals and investigating the causes of generating and the pattern of this. One of the most important hydrodynamic parameters of the canal is waves generated.

One of the most comprehensive simulation and prediction of waves can be achieved by using analytical methods, because the hydraulic simulation models, have computational limitations. So it limits possibility of applying in small areas.

When in canals, depth or flow rate is suddenly changed, the sharp wave is generated.

Mainly, sharp waves are generated in super critical flow, however flow from sub-critical to super critical is a sharp wave that is called hydraulic jump. If the location and characteristics of sharp wave changes with time, or sharp wave moves, it is called dynamic-sharp waves. For example, the generated waves in dam failure or sudden opening and closing of valves are dynamic-sharp waves. On the other side, if the position and characteristics of the sharp wave, after the steady flow, remain constant, it is called the static-sharp wave. So generated jump hydraulic and sharp wave in transitions due to cross-obstacle in flow path at supercritical flows are classified as the static sharp waves. The equations governing the dynamics of sharp waves are the same as the flow equation in shallow-water.

Theoretical aspects of the problem of small-amplitude water waves travelling in a region of varying depth, mainly uniqueness results, have been presented, under various geometric assumptions, by Vainberg & Maz'ja (1973), Fitz-Gerald (1976),Fitz-Gerald & Grimshaw (1979), Simon & Ursell (1984), Kuznetsov (1991, 1993) and other authors. See also the survey by Evans & Kuznetsov (1997). Besides, general methods for direct numerical solution of the linear problem, such as finite-element methods, boundary-integral-equation methods or hybrid techniques are available; see, e.g. the surveys by Mei (1978, 1983), Euvrard et al. (1981), Yeung (1982), Porter & Chamberlain (1997). Moreover, general numerical techniques based on a topography discretization or on a domain transformation have been developed. See, e.g. Devillard, Dunlop & Souillard (1988), Rey (1992) for the former, and Evans & Linton (1994) for the latter approach. However, the computational cost of these generic techniques is high, rendering them inappropriate, especially for long-range propagation and three dimensions. Owing to this fact, a constantly growing emphasis has been given on the development of approximate wave models retaining only the essential features of specific families of problems and, thus, being better suited for long-range wave propagation.

A well-known specific feature of water waves is that the propagation space does not coincide with the physical space. While the latter is the whole liquid domain (an irregularly shaped horizontal strip in the case of a shallow-sea environment), the former is only the horizontal direction(s). This fact, which is a manifestation of the surface (boundary) character of water waves, leads to the reformulation of the propagation problem as a non-local wave equation in the propagation (horizontal) space. In the linear case, on which we shall focus our attention in the present work, the appropriate non-local equation may take the form (in the time domain) of either an operator differential equation (Garipov 1965; Moiseev 1964; Moiseev&Rumiantsev1968; Craig & Sulem 1993), or an Integra differential equation (Fitz-Gerald 1976), or a pseudo differential equation (Milder 1977; Miles 1977; Craig &Sulem 1993) or even a partial differential equation with respect to the complex-analytic wave potential(Athanassoulis & Makrakis 1994). Another possibility, which will be discussed in detail in the present work, is to reformulate the problem as an infinite system of horizontal equations with variable coefficients. A clear consequence of the non-local character of the water-wave problem in the propagation space is that any one-equation model (i.e. one deferential equation in the horizontal direction(s)) cannot capture all features of the problem. Nevertheless, because of the complexity of the problem, a plethora of one-equation models have been proposed and studied, each one with its range of applicability and its advantages.

Historically, Eckart (1952) was the first author who proposed a one-equation model for intermediate-depth water. Breakoff (1972, 1976) derived a slightly more general model, called the mild-slope equation. Both authors used predetermined vertical distributions of the wave potential and applied a depth averaging procedure in order to obtain equations in the propagation space. Other derivations of similar or improve done-equation models, using either averaging techniques or variation principles, have been given by Smith & Sprinks (1975), Lozano & Meyer (1976), Booij (1981), Radder & Dingemans (1985), Kirby (1986a; b), Massel (1993); see also the general surveys by Massel (1989), Mei (1983), Porter & Chamberlain (1997) and Dingemans (1997)and the many references cited there, as well as the paper by Miles (1991), where a thorough comparison between Eckart's and Berkhoff's models is made. In general,mild-slope equations can be considered satisfactory for mean bottom slopes up to 1 : 3(Booij 1983; Berkhoff, Booij & Radder 1982) and some of them (the appropriately modified ones such as Kirby 1986a; b), can also predict the high backscattering due to Bragg resonance, occurring when an undulating component is super imposed on a slowly varying bottom topography. (Bragg scattering has also been studied by using perturbation techniques by Davies & Heathershaw 1984, Mei 1985 and Hara & Mei1987). Another important feature of most of the mild-slope models (but not Eckart's, Miles 1991) is that, despite their approximate character, they conserve wave energy.

According to past research, it can be said that different numerical methods are presented for solving the wave equation with different boundary conditions. In this paper the equation of shallow-water for solving wave linear equation is used.

2. Solving the wave equation in the shallow-water

To solve the wave equation shallow-water equations is used. Schematic view of a wave can be seen in Figure 1.

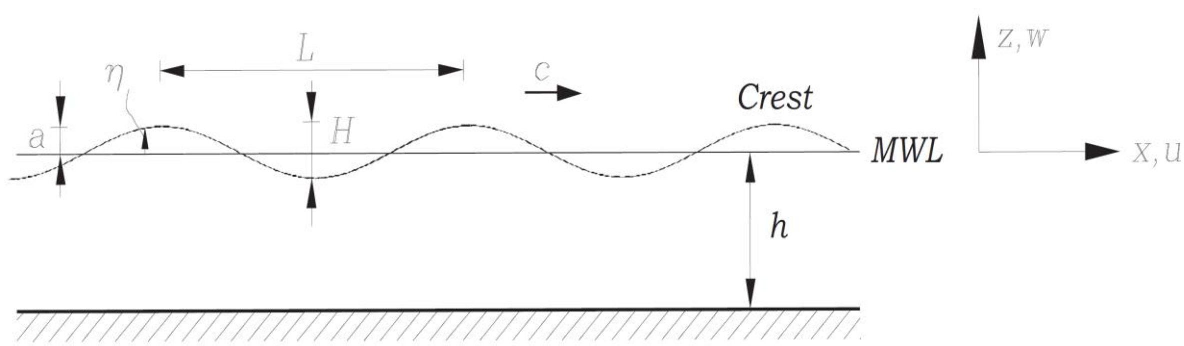

Fig. 1. Schematic view of a wave.

WhereH is wave height, a iswave amplitude, η is water surface elevations from MWL (positive upwards), L is wave length, c is velocity of wave, and h is water depth. To solve the Laplace equation in this paper all the solutions is done based on the axes specified in Figure 1.In all topics of wave, three parameters defined for waves: the wave number (k), angular frequency (w) and frequency (f). Which are defined below.

$$k = \frac{2\pi}{L}\ (rad/s) \qquad \text{Wave Number}$$

$$w = \frac{2\pi}{T}\ (rad/s) \qquad \text{Angular Frequency}$$

$$f = \frac{1}{T}\ (Hz,\ s^{-1}) \qquad \text{Frequency}$$

Where, T is wave period, time between two crests passage of same vertical section, can be said according to Figure 1, the upper surface of the wave is a function of the longitudinal distance (x) and time (t), that can be expressed as follows.

$$\eta_{(x,t)} = \mathrm{a}\sin\left(\frac{2\pi}{T}t - \frac{2\pi}{L}x\right) \tag{1}$$

In general, wave period and wave length, across the canal length can be proved as follows:

$$\eta_{(x,t)} = \eta_{(x+L,t)} = \eta_{(x+nL,t)} \qquad n = ...,\text{-}1,0,1,.....$$
$$\eta_{(x,t)} = \eta_{(x,t+T)} = \eta_{(x,t+nT)} \qquad n = ...,\text{-}1,0,1,..... \tag{2}$$

According to Equation (2)can be concluded:

$$\eta_{(x+x_0,t+t_0)} = \eta_{(x,t)} \quad \Rightarrow \quad \frac{x_0}{L} = \frac{t}{T}$$
$$-1 \le \sin(\mathrm{a}) \le 1 \;\rightarrow \left|\eta_{(x,t)}\right| \le \mathrm{a} \tag{3}$$

On the other side, for two-dimensional flow, velocity can be written as Equation (4):

$$\mathrm{V}_{(\mathrm{x,z.t})} = \mathrm{u}_{(\mathrm{x,z.t})}i + \mathrm{w}_{(\mathrm{x,z.t})}k \tag{4}$$

One of the governing equations is the Laplace equation, can be written in three-dimensional as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5}$$

According to Laplace's equation and potential velocity (ϕ), two-dimensional Laplace equation in terms of velocity potential can be written as follows:

$$\left|\begin{array}{l} u = \dfrac{\partial \phi}{\partial x} \\ v = \dfrac{\partial \phi}{\partial y} \\ w = \dfrac{\partial \phi}{\partial z} \end{array}\right. \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{Laplace Equation} \tag{6}$$

2.1. Boundary Condition

For this special conditions, bottom and surface boundary conditions can be expressed as follows.

$$\mathrm{w}_{(\mathrm{x,z=-h.t})} = \frac{\partial \varphi}{\partial \mathrm{z}}_{(\mathrm{x,z=-h.t})} = 0 \qquad \text{Bottom Boundary Condition} \tag{7}$$

$$\left|\begin{array}{l} \eta_{(\mathrm{x}_2,\mathrm{t}_2)} = \eta_{(\mathrm{x}_1,\mathrm{t}_1)} + \mathrm{w}(\mathrm{t}_2\text{-}\mathrm{t}_1) \\ \mathrm{x}_2 = \mathrm{x}_1 + \mathrm{u}(\mathrm{t}_2\text{-}\mathrm{t}_1) \end{array}\right. \qquad \text{Surface Boundary Condition} \tag{8}$$

According to the surface boundary conditions and Figure 2, Taylor series is used to obtain the value of $\eta_{(\mathrm{x}_2,\mathrm{t}_2)}$, according to the Equations, the Taylor series for the one-dimensional case (eq. 9) and two-dimensional (Eq. 10) is as follows:

$$f_{(x_0+\Delta x)} = f_{(x_0)} + \Delta x f'_{(x_0)} + \frac{\Delta x^2}{2!} f''_{(x_0)} + ... + \frac{\Delta x^n}{n!} f^{(n)}{}_{(x_0)} + R_{n(x_0+\Delta x)} \qquad \text{One Dimensional} \tag{9}$$

$$f_{(x_0+\Delta x,\, y_0+\Delta y)} = f_{(x_0,y_0)} + \Delta x \frac{\partial f_{(x_0,y_0)}}{\partial x} + \Delta y \frac{\partial f_{(x_0,y_0)}}{\partial y} + \frac{1}{2}\Delta x^2 \frac{\partial^2 f_{(x_0,y_0)}}{\partial x^2} +$$

$$\Delta x \Delta y \frac{\partial^2 f_{(x_0,y_0)}}{\partial x \partial y} + \frac{1}{2}\Delta y^2 \frac{\partial^2 f_{(x_0,y_0)}}{\partial y^2} + O_{(\Delta x^3)} \qquad \text{Two Dimensional} \tag{10}$$

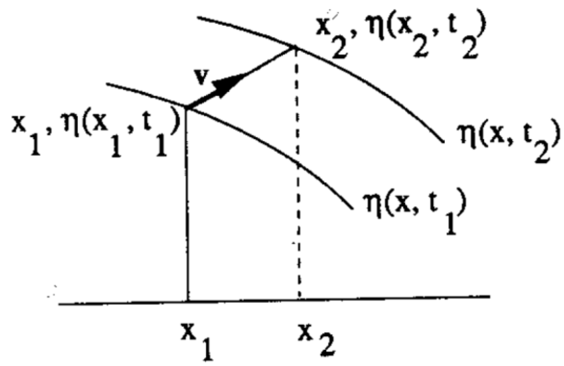

Fig. 2. Schematicview ofthe wave transmission.

Using two-dimensional Taylor series approximation (equation 10) for equation (8):

$$\eta_{(x_2,t_2)} = \eta_{(x_1+\Delta x, t_1+\Delta t)}$$

$$\Delta x = x_2 - x_1$$

$$\Delta t = t_2 - t_1$$

$$\eta_{(x_2,t_2)} = \eta_{(x_1+\Delta x, t_1+\Delta t)} = \eta_{(x_1,t_1)} + (x_2 - x_1)\frac{\partial \eta_{(x_1,t_1)}}{\partial x} + (t_2 - t_1)\frac{\partial \eta_{(x_1,t_1)}}{\partial t}$$

$$\eta_{(x_1,t_1)} + (x_2 - x_1)\frac{\partial \eta_{(x_1,t_1)}}{\partial x} + (t_2 - t_1)\frac{\partial \eta_{(x_1,t_1)}}{\partial t} = \eta_{(x_1,t_1)} + w(t_2 - t_1)$$

$$\Rightarrow \frac{\partial \eta_{(x_1,t_1)}}{\partial x}\frac{(x_2 - x_1)}{(t_2 - t_1)} + \frac{\partial \eta_{(x_1,t_1)}}{\partial t}\frac{(t_2 - t_1)}{(t_2 - t_1)} = w$$

$$\xrightarrow{u=\frac{(x_2-x_1)}{(t_2-t_1)}} u\frac{\partial \eta_{(x_1,t_1)}}{\partial x} + \frac{\partial \eta_{(x_1,t_1)}}{\partial t} = w \;\Rightarrow\; u\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} = w \qquad \text{Kinemetic Surface Boundary Condition} \tag{11}$$

Equation (11) is represented the kinematic boundary conditionson the wave. One of the other boundary conditions that are related to the wave, is dynamic boundary conditions at free surface.To obtain an equation for the dynamic boundary conditions, Bernoulli's equation is used that is one of the other governing equations. Bernoulli equation for steady and unsteady state according to Figure 3 is calculated as follows.

$$\sum F_s = ma_s \qquad \text{Newton's Second Law}$$

$$PdA-(P+dP)dA-w\sin\theta=ma_s \tag{12}$$

$$a_s = \frac{dv}{dt}$$

$$\mathrm{dv}=\frac{\partial \mathrm{v}}{\partial s}ds+\frac{\partial \mathrm{v}}{\partial t}dt \Rightarrow \frac{\mathrm{dv}}{dt}=\frac{\partial \mathrm{v}}{\partial s}\frac{ds}{dt}+\frac{\partial \mathrm{v}}{\partial t} \Rightarrow \frac{\mathrm{dv}}{dt}=\frac{\partial \mathrm{v}}{\partial s}v+\frac{\partial \mathrm{v}}{\partial t}$$

$$\text{Steady Flow} \quad \rightarrow \quad \frac{\partial \mathrm{v}}{\partial t}=0$$

$$\text{Unsteady Flow} \quad \rightarrow \quad \frac{\partial \mathrm{v}}{\partial t}\neq 0$$

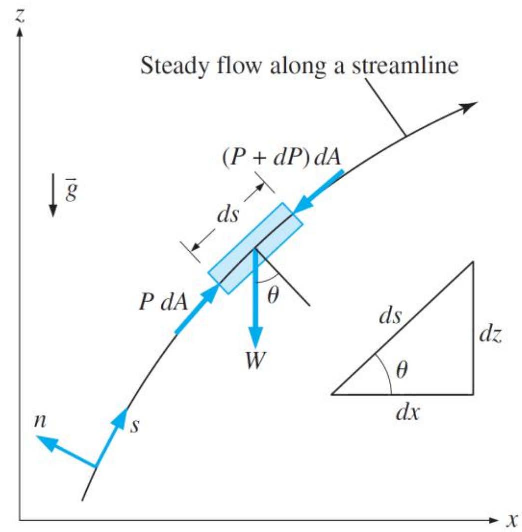

Fig. 3. Elements identified during the wave and forces acting on the element.

For unsteady flow, acceleration is calculated as follows:

$$a_s=\frac{\mathrm{dv}}{dt}=\frac{\partial \mathrm{v}}{\partial s}\frac{ds}{dt}+\frac{\partial \mathrm{v}}{\partial t} \Rightarrow \frac{\mathrm{dv}}{dt}=\frac{d\mathrm{v}}{ds}v+\frac{\partial \mathrm{v}}{\partial t} \tag{13}$$

Equation (14) is derived from equation (13) and equation (12) as follows:

$$\mathrm{PdA}-(\mathrm{P+dP})\mathrm{dA}-\mathrm{wsin}\theta=\mathrm{ma}_s=\mathrm{m}\left(v\frac{d\mathrm{v}}{ds}+\frac{\partial \mathrm{v}}{\partial t}\right)$$

$$\left|\begin{array}{l}\mathrm{w}=\rho\mathrm{gdAds}\\ \sin\theta=\dfrac{\mathrm{dz}}{\mathrm{ds}}\\ \mathrm{m}=\rho\mathrm{dAds}\end{array}\right. \Rightarrow \mathrm{PdA}-\mathrm{PdA}-\mathrm{dPdA}-\rho\mathrm{gdAds}\frac{\mathrm{dz}}{\mathrm{ds}}=\rho\mathrm{dAds}\left(v\frac{d\mathrm{v}}{ds}+\frac{\partial \mathrm{v}}{\partial t}\right)$$

$$\Rightarrow -\mathrm{dP}-\rho\mathrm{gdz}=\rho v d\mathrm{v}+\rho ds\frac{\partial \mathrm{v}}{\partial t}$$

$$\Rightarrow -\frac{\mathrm{dP}}{\rho}-\mathrm{gdz}=\frac{1}{2}d(\mathrm{v}^2)+\frac{\partial \mathrm{v}}{\partial t}ds$$

Integrating the above equation:

$$\Rightarrow \int \frac{\mathrm{dP}}{\rho} + \int \frac{\partial \mathrm{v}}{\partial t} ds + \frac{\mathrm{v}^2}{2} + gz = \text{Constant}$$

$$\int \frac{\partial \mathrm{v}}{\partial t} ds = \frac{\partial \phi}{\partial t}$$

$$\mathrm{v}^2 = \mathrm{u}^2 - \mathrm{w}^2$$

$$\Rightarrow \frac{\mathrm{P}}{\rho} + \frac{\partial \phi}{\partial t} + \frac{\left(\mathrm{u}^2 - \mathrm{w}^2\right)}{2} + gz = C_{(t)} \quad (14)$$

Equation (14) is known the Bernoulli equation unsteady flow, now by considering to the water surface boundary conditions $C_{(t)} = \frac{\mathrm{P_{atm}}}{\rho}$:

$$\frac{P}{\rho} + \frac{\partial \phi}{\partial t} + \frac{\left(u^2 - w^2\right)}{2} + g\eta = C_{(t)} \qquad \text{Unsteady Bernouli Equation}$$

$$\xrightarrow{C_{(t)} = \frac{P_{atm}}{\rho}} \frac{P}{\rho} + \frac{\partial \phi}{\partial t} + \frac{\left(u^2 - w^2\right)}{2} + g\eta = \frac{P_{atm}}{\rho}$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + \frac{\left(u^2 - w^2\right)}{2} + g\eta = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 - \left(\frac{\partial \phi}{\partial z}\right)^2\right) + g\eta = 0 \qquad \text{Dynamic Boundary Condition} \quad (15)$$

Equation (15) represents the dynamic boundary conditions. Considering that wave is periodic form, it needs a periodic condition to be defined for wave. The periodic conditions based on Fig. 4 and eq. (2) is defined as follows.

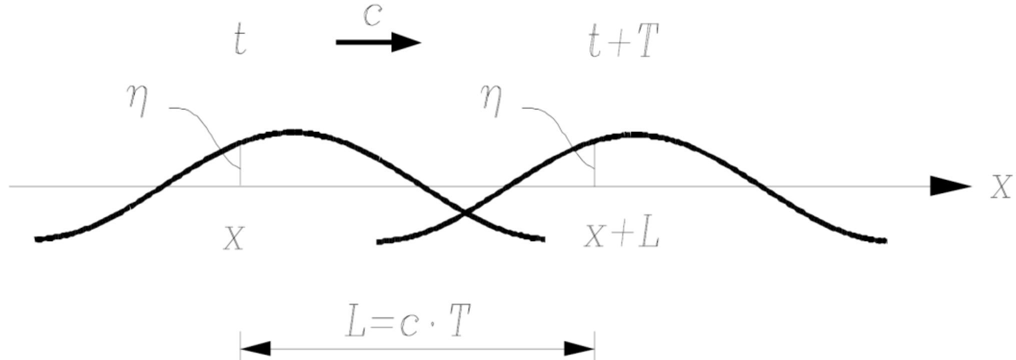

Figure 4.Schematicviewof thefrequency ofawave

$$\eta_{(x,t)} = \eta_{(x+nL,t)} = \eta_{(x,t+nT)} \qquad n = 1, 2, 3, \ldots$$

$$(x,t) \to (L\frac{t}{T} - x) \quad \Rightarrow \eta_{\left(L\frac{t}{T}-x\right)} = \eta_{\left(L\frac{(t+nT)}{T}-(x+nL)\right)}$$

Dimensionless parameter are as follows:

$$\text{Dimensionless} \quad \rightarrow \quad \frac{2\pi}{L}\left(L\frac{t}{T} - x\right) = 2\pi\left(\frac{t}{T} - \frac{x}{L}\right)$$

$$\rightarrow \begin{cases} \eta_{(x,t)} = \eta_{(\theta)} \\ \phi_{(x,z.t)} = \phi_{(\theta,z)} \end{cases} \rightarrow \quad \theta = 2\pi\left(\frac{t}{T} - \frac{x}{L}\right)$$

From the above equations the parameter θ is replaced by the parameters x and t, as can be seen.Hence, according to the wave number (k) and angular velocity (w),θ parameter can be defined as follows:

$$\begin{cases} \mathrm{k}=\dfrac{2\pi}{L} \\ w=\dfrac{2\pi}{T} \end{cases} \rightarrow \quad \theta=\mathrm{wt}-\mathrm{kx} \tag{16}$$

Now linear equations for the boundary conditions can be revealed as the following.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \Rightarrow \quad \nabla\phi = 0 \qquad \text{for} \quad -\infty < x < \infty \,,\ -\mathrm{h} < z < 0 \qquad \text{Laplace Equation} \tag{17}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad at \quad z = -h \qquad \text{Bottom Boundary Condition} \tag{18}$$

$$\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial \mathrm{x}} + \frac{\partial\eta}{\partial \mathrm{t}} = \frac{\partial\phi}{\partial \mathrm{z}} \quad \Rightarrow \quad \frac{\partial\eta}{\partial \mathrm{t}} = \frac{\partial\phi}{\partial \mathrm{z}} \qquad \text{at} \quad \mathrm{z}=0 \qquad \text{Kinemetic Surface Boundary Condition} \tag{19}$$

$$\frac{\partial\phi}{\partial t} + g\eta = 0 \qquad \text{at} \qquad \mathrm{z}=0 \qquad \text{Dynamic Boundary Condition} \tag{20}$$

$$\eta_{(\mathrm{x,t})} \ \text{and} \ \phi_{(\mathrm{x,z.t})} \rightarrow \eta_{(\theta)} \ \text{and} \ \phi_{(\theta,\mathrm{z})} \quad \text{that} \quad \theta=\mathrm{wt\text{-}kx} \qquad \text{Periodicity Boundary Condition} \tag{21}$$

According to the above equations, here we have two surface boundary conditions for the wave, kinematic conditions (Equation19) and dynamic conditions (Equation20). The integration of these two equations can be obtained from the following general equation (Equation22) for the surface boundary conditions.

$$\rightarrow \frac{\partial\phi}{\partial t} + g\eta = 0 \quad \Rightarrow \quad \frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\eta}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial\eta}{\partial t} = -\frac{1}{g}\frac{\partial^2\phi}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial z} = -\frac{1}{g}\frac{\partial^2\phi}{\partial t^2} \quad \Rightarrow \quad \frac{\partial\phi}{\partial z} + \frac{1}{g}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{22}$$

2.2. Boundary conditions based on frequency

Since one of the main characteristics of the wave is frequency, the periodic boundary conditions entered at the other boundary conditions and the governing equations(equations 20-17)and the wave equation solved with this equations. The conversion process is shown in the following equations.

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\theta}.\frac{\partial\theta}{\partial x} = \frac{\partial\phi}{\partial\theta}(-k)$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = \frac{\partial\left(\frac{\partial\phi}{\partial x}\right)}{\partial x} = \frac{\partial\left(\frac{\partial\phi}{\partial x}\right)}{\partial\theta}.\frac{\partial\theta}{\partial x} = \frac{\partial\left(\frac{\partial\phi}{\partial\theta}(-k)\right)}{\partial\theta}(-k) \quad \Rightarrow \quad k^2\frac{\partial^2\phi}{\partial\theta^2} = \frac{\partial^2\phi}{\partial x^2}$$

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial\theta}.\frac{\partial\theta}{\partial t} = w\frac{\partial\phi}{\partial\theta} \qquad \Rightarrow \qquad \frac{\partial^2\phi}{\partial t^2} = w^2\frac{\partial^2\phi}{\partial\theta^2}$$

$$k^2 \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{Laplac Equation} \tag{23}$$

$$\frac{\partial \phi}{\partial z} + \frac{w^2}{g} \frac{\partial^2 \phi}{\partial \theta^2} = 0 \qquad \text{Combination of Surface Boundary Condition} \tag{24}$$

According to the following boundary conditions have:

$$\text{For} \begin{cases} \text{x=0, t=t} \rightarrow \theta = 2\pi \frac{\text{t}}{\text{T}} \\ \text{x=L, t=t} \rightarrow \theta = 2\pi \frac{\text{t}}{\text{T}} - 2\pi \end{cases} \Rightarrow -k \frac{\partial \phi}{\partial \theta}_{\left(2\pi \frac{\text{t}}{\text{T}}, z\right)} = -k \frac{\partial \phi}{\partial \theta}_{\left(2\pi \frac{\text{t}}{\text{T}} - 2\pi, z\right)}$$

$$\text{For} \begin{cases} \text{x=-L} \\ \text{t=0} \end{cases} \rightarrow -k \frac{\partial \phi}{\partial \theta}_{(0,z)} = -k \frac{\partial \phi}{\partial \theta}_{(2\pi,z)} \qquad \text{Periodicity Boundary Condition} \tag{25}$$

2.3. Generate and solve differential equations

Now according to the potential velocity parameter, (ϕ) ,a function of X and z, The equation of velocity potential can be written as equation (26), and then according to the Laplace equation (Equation 23) differential equations for the solution of wave can be achieved.

$$\phi_{(\theta,z)} = f_{(\theta)}.Z_{(z)} \tag{26}$$

$$\text{Laplac Equation} \rightarrow k^2 f''_{(\theta)} Z_{(z)} + f_{(\theta)} Z''_{(z)} = 0$$

$$\xrightarrow{/ f_{(\theta)}.Z_{(z)}} \quad -k^2 \frac{f''_{(\theta)}}{f_{(\theta)}} = \frac{Z''_{(z)}}{Z_{(z)}}$$

$$-k^2 \frac{f''_{(\theta)}}{f_{(\theta)}} = \frac{Z''_{(z)}}{Z_{(z)}} = \lambda^2 \rightarrow \begin{cases} f'' + f(\frac{\lambda^2}{k^2}) = 0 \\ Z'' - Z(\lambda^2) = 0 \end{cases} \qquad \text{Differential Equation} \tag{27}$$

With solving these equations (Equation 27), and the effect of boundary conditions, the wave equation can be derived.To solve mentioned equations, the Euler equation expressed as follows, are used.

$$\text{L}[\text{y}] = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y = r(x) \qquad \text{Euler Equation}$$

$$y = e^{rx} \rightarrow r^n + a_1 r^{n-1} + \ldots + a_n = 0$$

$$\text{The Solution of this is} \rightarrow r_1, r_2, \ldots, r_n$$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x} \tag{28}$$

$$\text{The Solution of this is} \rightarrow \alpha + \beta i \ , \ \alpha - \beta i$$

$$y = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x}$$

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$\Rightarrow \quad y = e^{\alpha x}\left(c_1 \cos \beta x + c_2 \sin \beta x\right) \tag{29}$$

Now by considering equations (28) and (29), presented by Euler, Equations (27) can be solved as follows.

$$f'' + f(\frac{\lambda^2}{k^2}) = 0$$

$$\xrightarrow{\text{Euler Equation}} \mathrm{r}^2 + \frac{\lambda^2}{k^2} = 0 \quad \rightarrow \quad \mathrm{r}^2 = -\frac{\lambda^2}{k^2} \quad \rightarrow \quad \mathrm{r} = \pm\frac{\lambda}{k} i$$

$$\Rightarrow \quad f = A_1 e^{\left(\frac{\lambda}{k} i\right) f} + A_2 e^{\left(-\frac{\lambda}{k} i\right) f}$$

$$\Rightarrow \quad f = A_1 \cos\left(\frac{\lambda}{k}\theta\right) + A_2 \sin\left(\frac{\lambda}{k}\theta\right) = A\sin\left(\frac{\lambda}{k}\theta + \delta\right)$$

$$in \quad \left| \begin{array}{l} \theta_{(x,t)} \rightarrow \delta = 0 \\ x = L \rightarrow \dfrac{\lambda}{k} = 1 \end{array} \right. \quad \Rightarrow \quad f = A\sin(\theta) \tag{30}$$

$$Z'' - Z(\lambda^2) = 0$$

$$\xrightarrow{\text{Euler Equation}} \mathrm{r}^2 - \lambda^2 = 0 \quad \rightarrow \quad \mathrm{r}^2 = \lambda^2 \quad \rightarrow \quad \left| \begin{array}{l} r = -\lambda \\ r = +\lambda \end{array} \right.$$

$$\Rightarrow \quad Z = B_1 e^{\lambda z} + C_1 e^{-\lambda z}$$

$$\left| \begin{array}{l} \sinh x = \dfrac{e^x - e^{-x}}{2} \\ \cosh x = \dfrac{e^x + e^{-x}}{2} \end{array} \right. \quad \Rightarrow \quad \left| \begin{array}{l} B_1 = \dfrac{B+C}{2} \\ C_1 = \dfrac{B-C}{2} \\ \lambda = k \end{array} \right. \quad \Rightarrow \quad Z = B\cosh kz + C\sinh kz \tag{31}$$

Equation (30) and (31) are derived from differential equations (Equation 27). Three values of the constants A, B and C are obtained from the bottom and surface boundary conditions.

2.4. Derived the constant parameters

According to equation (26) and bottom boundary conditions (Equation18) constant B can be omitted from the equation (31) and the equation can be converted to equation (32) as the following.

$$Z = B\cosh(kz) + C\sinh(kz) \quad \rightarrow \quad Z'_{(-h)} = Bk\sinh(-kh) + Ck\cosh(-kh)$$

$$\xrightarrow{Z'_{(-h)}=0} \quad \mathrm{B} = C\coth(kh)$$

$$\Rightarrow Z = C\left(\coth(kh).\cosh(kz) + \sinh(kz)\right)$$

$$\Rightarrow Z = \frac{C}{\sinh(kh)}\left(\cosh(kh).\cosh(kz) + \sinh(kh)\sinh(kz)\right)$$

$$\Rightarrow Z = C\frac{\cosh k(z+h)}{\sinh kh} \tag{32}$$

According to equation (32) and (30), equation (26) can be written as follows:

$$\left| \begin{array}{l} \phi_{(\theta,z)} = f_{(\theta)}.Z_{(z)} \\ f = A\sin(\theta) \\ Z = C\dfrac{\cosh k(z+h)}{\sinh kh} \end{array} \right. \rightarrow \quad \phi = A.C\frac{\cosh k(z+h)}{\sinh kh}\sin(\theta) \tag{33}$$

To derive the constants A and C from the surface dynamic boundary condition (Equation20) the following can be done.

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at} \quad z=0 \quad \rightarrow \quad \eta = -\frac{1}{g}\frac{\partial \phi}{\partial t} \quad \text{at} \quad z=0$$

$$\phi = A.C\frac{\cosh k(z+h)}{\sinh kh}\sin\theta \qquad \text{That} \quad \theta = wt - kx$$

$$\frac{\partial \phi}{\partial t} = A.C.w\frac{\cosh k(z+h)}{\sinh kh}\cos\theta$$

$$\rightarrow \eta = -\frac{w}{g}.A.C\frac{\cosh kh}{\sinh kh}\cos\theta \quad \text{at} \quad z=0$$

$$\eta = \mathrm{a}\cos\theta \tag{34}$$

$$\Rightarrow \mathrm{a} = -\frac{w}{g}.A.C\frac{\cosh kh}{\sinh kh} \quad \rightarrow \quad \mathrm{A.C} = \frac{\mathrm{a}.g}{-w}\frac{\sinh kh}{\cosh kh} \tag{35}$$

$$\left| \begin{array}{l} \phi = A.C\dfrac{\cosh k(z+h)}{\sinh kh}\sin\theta \\ \mathrm{A.C} = \dfrac{\mathrm{a}.g}{-w}\dfrac{\sinh kh}{\cosh kh} \\ \theta = wt - kx \end{array} \right. \Rightarrow \quad \phi = \frac{\mathrm{a}.g}{-w}\frac{\cosh k(z+h)}{\cosh kh}\sin(wt - kx) \tag{36}$$

According to the mentioned solution, equation (34) and equation (36) is derived for static sharp waves in shallow-water.

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